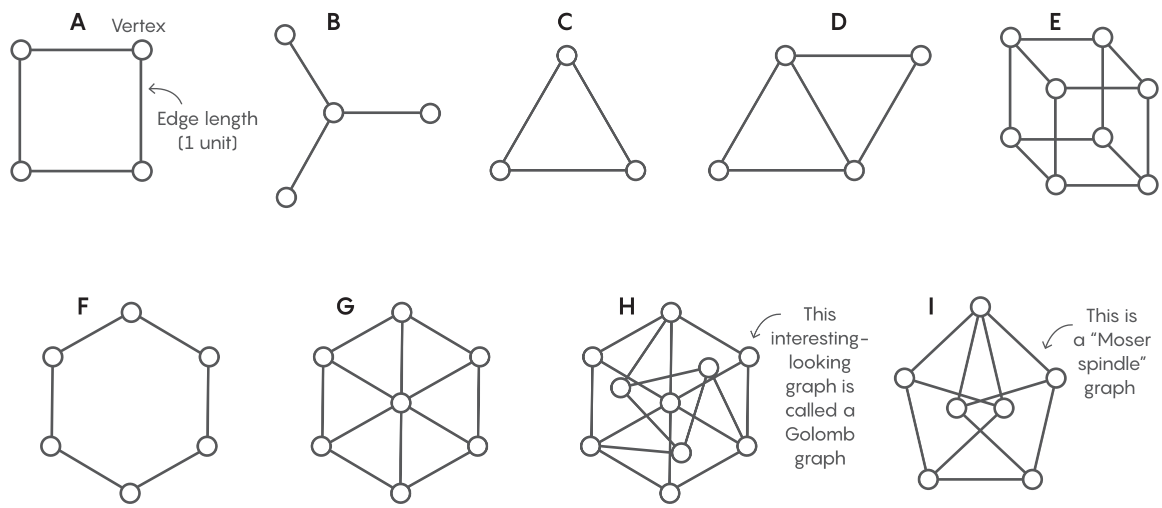
Mathematical Coloring

In geometric graph theory, the Hadwiger–Nelson problem asks for the smallest number of colors needed to color a plane so that no two points 1 unit apart are the same color. The exact "chromatic number of the plane" is still unknown, but mathematicians have established a lower and upper bound.

FINDING A LOWER BOUND

The unit-distance graphs to the right, which all have edges (lines) of 1 unit, can serve as a simplification of the plane to help us find a lower bound. Color the vertices (nodes) so that no two connected vertices are the same color, thereby satisfying the condition of the Hadwiger-Nelson problem.

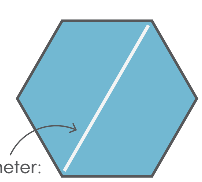


Q1 What is the minimum number of colors necessary to color each unit-distance graph so that no two adjacent vertices are the same color?

FINDING AN UPPER BOUND

Now consider an infinite expanse of points on a plane.
We'll use hexagon tilings to find an upper bound for the Hadwiger-Nelson problem.

1. The diameter of this hexagon is less than 1 unit, so all of the blue points within it are less than 1 unit apart.

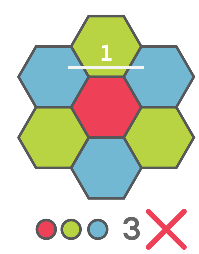


2. The "infinite plane" is a little overwhelming, so let's consider a small section:

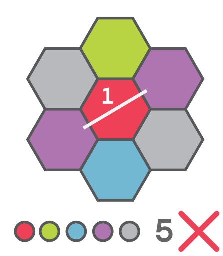
7 hexagons

3. We can see that three colors will not be enough to color this collection of seven hexagons:

Since the diameter of the hexagon is $d < 1$, the edge length is $d/2 < 1/2$ (to see why, divide the hexagon into six equilateral triangles). You can find two points near either end of an edge that are 1 unit apart and the same color.



It's also easy to find two points of the same color that are 1 unit apart when five colors are available:



Q2 We have shown that five colors are not enough to successfully color the collection of seven hexagons. What is the minimum number of colors required?

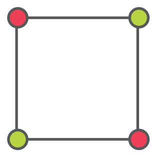
Quanta magazine

ANSWER KEY

A1 We need at least four colors to color all the graphs.

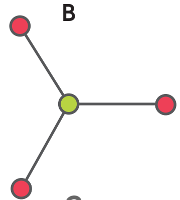
(Tip: For 60 years, mathematicians could not find a lower bound higher than four. But in April 2018, the biologist Aubrey de Grey published a unit-distance graph that requires five colors.)

A



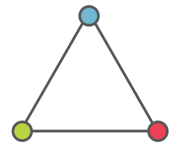
2 colors are needed

B



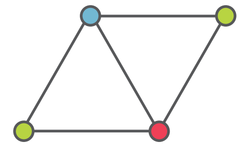
2 colors are needed

C



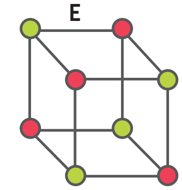
We need at least 3

D



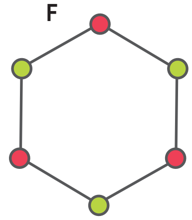
We need at least 3

E



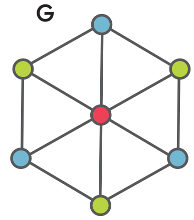
2 colors are needed

F



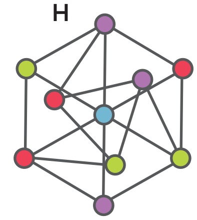
We need at least 2

G



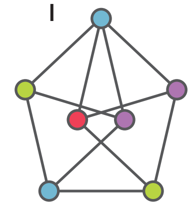
We need at least 3

H



We need at least 4

I

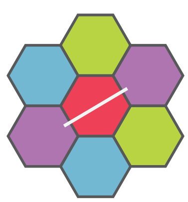


We need at least 4

Correction June 19, 2018: A previous iteration of this worksheet incorrectly showed that the cube graph (E) required at least 3 colors. The cube graph requires only 2 colors.

A2 We can successfully color our collection of seven hexagons using seven colors.

3 ✗



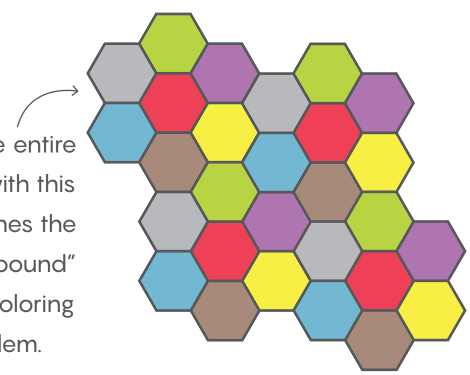
4 ✗

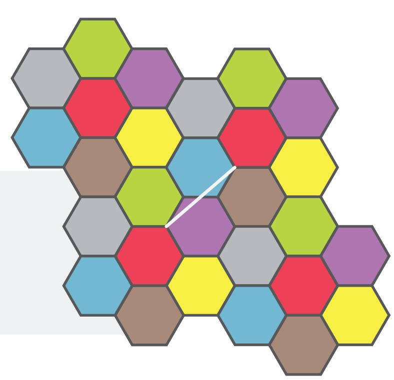
5 ✗

6 ✗

7 ✓

Tiling the entire plane with this establishes the "upper bound" of our coloring problem.





BONUS QUESTION:

Assuming the diameter of each hexagon is *d*, find the distance between the two red hexagons in terms of *d*, as shown. How can we be sure this distance is greater than 1?